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THEORETICAL INVESTIGATION OF THE THERMAL ENTRANCE REGION IN STEADY, AXIALLY SYMMETRICAL SLUG FLOW WITH MIXED BOUNDARY CONDITIONS

S. GOŁOS

Department of Reactor Engineering, Institute of Nuclear Research, Świerk, near Warsaw, Poland

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Abstract—This paper is devoted to the theoretical analysis of the growth of the thermal boundary layer and the heat-transfer problem in a steady, axially symmetrical slug flow of an incompressible, isotropic medium with constant physical properties, without internal heat sources assuming mixed boundary conditions.

The exact solution of this problem is obtained by use of the Laplace transformation.

The same problem is solved for comparative purposes, in an approximate way, by applying the principle of restricted variation.

The computations of the temperature field are carried out for liquid sodium flow in the case of Biot's number $Bi = 0{\cdot}5$ and Péclét's number $Pe = 22600$ using the exact and the approximate solution. The numerical results obtained enable us to draw the conclusion that the principle of restricted variation can be applied in a case, in which an exact solution is not possible to find.

NOMENCLATURE

- c, specific heat of the flowing medium;
- I, modified Bessel function of the first kind;
- J_r, restricted functional;
- J, Bessel function of the first kind;
- K, modified Bessel function of the second kind;
- m^2, auxiliary notation in equation (6), $(Pe\,.\,s)$;
- q, dimensionless radius of the thermal boundary layer;
- r, radius;
- R, outer radius of the flowing medium;
- s, complex variable in the Laplace transformation;
- T, temperature of the flowing medium;
- T_s, solidification temperature of the flowing medium;
- T_a, ambient temperature;
- T_e, evaporation temperature of the flowing medium;
- T_0, initial temperature of the flowing medium;
- u, auxiliary function in equation (6), $[Pe\,.\,(s\,.\,\bar{\theta} - 1)]$;
- $\mathbf{w}$, velocity vector of the flow;
- x, distance in flow direction measured from the end of insulated section;
- X_r, absolute length of the thermal entrance region, $(\xi_r\,.\,R)$;
- z, auxiliary notation in equation (6), $(m\,.\,\rho)$;
- Bi, Biot's number, $(R\,.\,\alpha/\lambda)$;
- Pe, Péclét's number, $(R\,.\,w/\varkappa)$.

Greek symbols

- α, heat-transfer coefficient;
- γ, specific weight of the flowing medium;
- ∇^2, Laplace operator;
- ∇, Hamilton's operator;
- δJ_r, restricted variation of the functional J_r;

θ, dimensionless temperature parameter, $\frac{T - T_a}{T_0 - T_a}$;
$\bar{\theta}$, Laplace transform of θ, $\mathscr{L}[\theta]$;
θ_1, approximate distribution of the dimensionless temperature parameter θ for the first phase of the approximate solution process;
θ_2, approximate distribution of the dimensionless temperature parameter θ for the second phase of the approximate solution process;
$\varkappa$, thermal diffusivity of the flowing medium, $(\lambda/c\,.\,\gamma)$;
λ, thermal conductivity of the flowing medium;
ξ, dimensionless distance in the direction of flow, measured from the end of insulated section (x/R);
ξ_r, dimensionless length of the thermal entrance region;
ρ, dimensionless radius, (r/R).

Subscripts
a, ambient;
e, evaporation;
r, entrance region; restricted;
s, solidification;
0, initial state; order of the Bessel function;
1, first phase of the approximate solution process; order of the Bessel function;
2, second phase of the approximate solution process.

1. INTRODUCTION

THE PROBLEM of determining of the temperature field in a medium flowing through a duct (for example a pipe) is of great importance for many branches of industry, but it arises with particular sharpness in fast reactors.

The complicated geometry of channels between fuel rods excludes beforehand the possibility of obtaining an exact solution determining the temperature field in the coolant flowing between these rods. This circumstance leads to the application of approximate methods. In this work the principle of restricted variation [1, 2] is used, our aim being to compare the results obtained by use of this principle with the exact solution. The problem of determining the temperature field in a steady axially symmetrical slug flow with mixed type boundary conditions has been chosen as an example. This case differs essentially from those cited in the scientific literature. The solution of this problem is encountered most often either in plane geometry with Dirichlet's boundary conditions [3, 4] or in cylindrical geometry with Neumann's boundary conditions [5].

2. EXACT SOLUTION

Our considerations concern the problem of determining the temperature field in an isotropic medium with constant physical properties, without internal heat sources, flowing with a constant velocity through an infinite thin-walled pipe.

The wall of this pipe is insulated on part of its length and the temperature of the medium flowing through the insulated part is everywhere constant, whereas on the uninsulated part of the wall the boundary condition is of the mixed type.

The thickness of the wall is so small, that its thermal resistance can be neglected and its temperature can be assumed as equal to the temperature of the medium flowing in the vicinity of the wall. The system considered is illustrated in Fig. 1.

The equation describing the temperature distribution in the interval $0 \leqslant r \leqslant R$ and $0 \leqslant x \leqslant +\infty$ can be written in the vector form as follows:

$$\lambda\,.\,\nabla^2 T - c\,.\,\gamma\,.\,(\boldsymbol{w}\nabla)\,T = 0. \qquad (1)$$

Taking into account the axial symmetry of the system and the slug flow of the medium as well as neglecting the conduction of heat in the

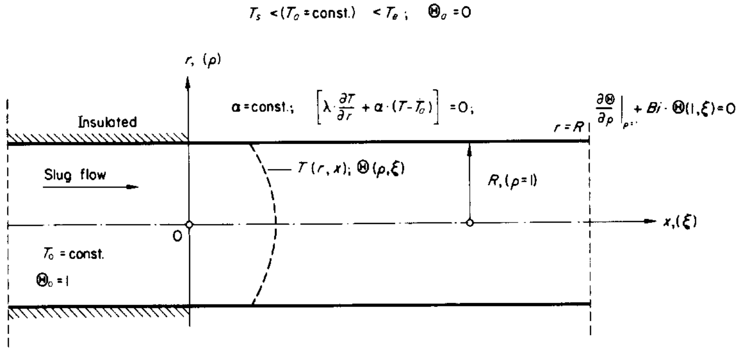

FIG. 1. Idealized model of the system under consideration.

direction of x, we can write equation (1) in the form:

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial T}{\partial r}\right) - \frac{w^{*}}{\varkappa} \cdot \frac{\partial T}{\partial x} = 0. \tag{2}$$

The boundary conditions for this equation are as follows:

$$\left.\begin{array}{ll} T(r, 0) = T_0; & T(r, +\infty) = T_a; \\ T_s < T_0 < T_e; & T_s < T_a < T_e; \\ \left.\dfrac{\partial T}{\partial r}\right|_{r=0} = 0; & \\ \left[\lambda \cdot \dfrac{\partial T}{\partial r} + \alpha \cdot (T - T_a)\right]_{r=R} = 0. & \end{array}\right\} \tag{3}$$

On expanding the term

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial T}{\partial r}\right)$$

and passing to the dimensionless quantities, we obtain for the equation (2):

$$\frac{\partial^2 \theta}{\partial \rho^2} + \frac{1}{\rho} \cdot \frac{\partial \theta}{\partial \rho} - Pe \cdot \frac{\partial \theta}{\partial \xi} = 0\,; \tag{4}$$

and, for the boundary conditions (3),

$$\left.\begin{array}{ll} \theta(\rho, 0) = 1\,; & \theta(\rho, +\infty) = 0\,; \\ \left.\dfrac{\partial \theta}{\partial \rho}\right|_{\rho=0} = 0; & \left[\dfrac{\partial \theta}{\partial \rho} + Bi \cdot \theta\right]_{\rho=1} = 0. \end{array}\right\} \tag{5}$$

The application of the Laplace transformation makes it possible to write the equation (4) in the form:

$$\frac{d^2 \bar{\theta}}{d\rho^2} + \frac{1}{\rho} \cdot \frac{d\bar{\theta}}{d\rho} - Pe \cdot (s \cdot \bar{\theta} - 1) = 0. \tag{6}$$

The boundary conditions for above equation are:

$$\left.\frac{d\bar{\theta}}{d\rho}\right|_{\rho=0} = 0; \quad \left[\frac{d\bar{\theta}}{d\rho} + Bi \cdot \bar{\theta}\right]_{\rho=1} = 0. \tag{7}$$

The introduction of an auxiliary function u and notations m^2 and z defined in the nomenclature reduces the equation (6) to the form:

$$\frac{d^2 u}{dz^2} + \frac{1}{z} \cdot \frac{du}{dz} - u = 0. \tag{8}$$

The general solution of (8) is:

$$u = C_1 \cdot I_0(z) + C_2 \cdot K_0(z); \tag{9}$$

where C_1 and C_2 are integration constants.

On returning to the primary function $\bar{\theta}$ and finding the integration constants from the boundary conditions (7), we obtain:

$$\bar{\theta} = \frac{1}{s} \cdot \left[1 + \right. \\ \left. - \frac{Bi . I_0[\sqrt{(Pe . s)} . \rho]}{\sqrt{(Pe . s)} . I_1[\sqrt{(Pe . s)}] + Bi . I_0[\sqrt{(Pe . s)}]}\right]. \tag{10}$$

The inverse Laplace transformation of the above expression yields (see Appendix):

$$\theta = 2 \, . \, Bi \, . \sum_{n=1}^{\infty} \frac{J_0(\zeta_n \, . \, \rho)}{(Bi^2 + \zeta_n^2) \, . \, J_0(\zeta_n)} \times \exp\left(-\frac{\zeta_n^2}{Pe} \, . \, \xi\right); \quad (11)$$

where numbers ζ_n are the roots of the following transcendental equation:

$$\zeta_n \, . \, J_1(\zeta_n) - Bi \, . \, J_0(\zeta_n) = 0. \quad (12)$$

The relation (11) is the exact solution of the equation (4) with the boundary conditions (5).

3. APPROXIMATE SOLUTION PROCESS BASED ON THE RESTRICTED VARIATION PRINCIPLE

The search for an approximate solution is based on the known idea of the thermal entrance region. The model of this idea for the case under consideration is described by Fig. 2.

thermal entrance region ξ_r. In other words, the first phase concerns the interval $0 \leqslant \xi \leqslant \xi_r$, whereas the second phase concerns the interval $\xi_r \leqslant \xi \leqslant +\infty$. The division of considerations into the two phases results from the fact, that the form of the function assumed as an approximate solution for the thermal entrance region must be different from the form of the function assumed for the interval outside this region.

3.1 *The first phase* ($0 \leqslant \xi \leqslant \xi_r$)

The starting relation is the equation (4) written in the self-adjoint form as follows:

$$\frac{\partial}{\partial \rho}\left(\rho \, . \, \frac{\partial \theta}{\partial \rho}\right) - \rho \, . \, Pe \, . \, X = 0; \quad (13)$$

where $X = \partial\theta/\partial\xi$.

In the present phase of the considerations the above equation is treated in the interval:

$$0 \leqslant \xi \leqslant \xi_r; \qquad q(\xi) \leqslant \rho \leqslant 1. \quad (14)$$

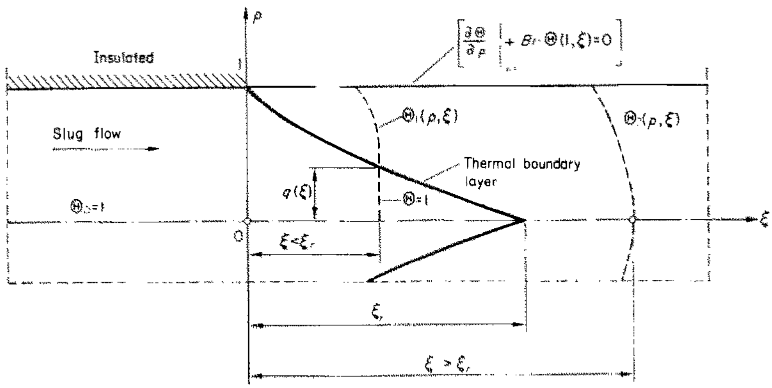

FIG. 2. Model of the thermal entrance region.

Our considerations will be divided into two phases. The first phase includes the thermal entrance region, in which the temperature penetration has not yet reached the axis of the flow. The distance from the origin of the coordinates to the place, where the temperature penetration has reached the axis of the flow, is defined as the dimensionless length of the

The boundary conditions belonging to the equation (13) are:

$$\left[\frac{\partial \theta}{\partial \rho} + Bi \, . \, \theta\right]_{\rho=1} = 0; \quad (15)$$

and

$$\left.\frac{\partial \theta}{\partial \rho}\right|_{\rho=q(\xi)} = 0; \qquad \left.\theta\right|_{\rho=q(\xi)} = 1. \quad (16)$$

The restricted variation principle allows us to write for equation (13) the restricted functional in the form:

$$J_r\langle\theta\rangle = \int_{q(\xi)}^{1} \left[\frac{1}{2}\cdot\left(\frac{\partial\theta}{\partial\rho}\right)^2 + Pe\,.\,X\,.\,\theta\right].\,\rho\,.\,\mathrm{d}\rho + \tfrac{1}{2}\,.\,Bi\,.\,\theta^2 = \text{Extremum}. \tag{17}$$

The first variation of the above functional is:

$$\delta J_r = -\int_{q(\xi)}^{1}\left[\frac{\partial}{\partial\rho}\left(\rho\,.\,\frac{\partial\theta}{\partial\rho}\right) - \rho\,.\,Pe\,.\,X\right].\,\mathrm{d}\rho\,.\,\delta\theta + \left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=q(\xi)}.\,\delta\theta - \left[\frac{\partial\theta}{\partial\rho} + Bi\,.\,\theta\right]_{\rho=1}.\,\delta\theta = 0. \tag{18}$$

An approximate solution satisfying the boundary conditions (15) and (16) can be written as follows:

$$\theta_1 = 1 - \frac{Bi\,.\,[1-q(\xi)]}{2 + Bi\,.\,[1-q(\xi)]} \times \left[1 - \frac{1-\rho}{1-q(\xi)}\right]^2. \tag{19}$$

The only unknown quantity in (19) is the parameter $q(\xi)$. In further considerations the parameter $q(\xi)$ is written, for brevity, as q, but it is constantly understood as a function of ξ. According to (18) we have:

$$\int_q^1 \left[\frac{\partial}{\partial\rho}\left(\rho\,.\,\frac{\partial\theta_1}{\partial\rho}\right) - \rho\,.\,Pe\,.\,\frac{\partial\theta_1}{\partial\xi}\right].\mathrm{d}\rho\,.\,\delta\theta_1 = 0. \tag{20}$$

Evaluation of the above integral, based on (19), yields an ordinary differential equation of the first order, determining the parameter q. This equation is of the form:

$$\frac{Pe\,.\,(B_1 - B_2\,.\,q + B_3\,.\,q^2 - B_4\,.\,q^3 + B_5\,.\,q^4 + B_6\,.\,q^5 - B_7\,.\,q^6 + B_8\,.\,q^7 - B_9\,.\,q^8)}{10\,.\,(1-q)\,.\,[2 + Bi\,.\,(1-q)]}\cdot\frac{\mathrm{d}q}{\mathrm{d}\xi} + (A_1 - A_2\,.\,q + A_3\,.\,q^2 - A_4\,.\,q^3 + A_5\,.\,q^4) = 0; \tag{21}$$

The thermal entrance region ends, where the temperature penetration has reached the axis of the flow ($q = 0$; see Fig. 2). The expression (24) makes it possible to evaluate the dimensionless length ξ_r. This length is determined by the formula:

$$\xi_r = \frac{Pe}{10}\cdot\left\{\frac{1}{2}\cdot N_1 + \frac{1}{3}\cdot N_2 + N_3 + \ln\left[\left(\frac{4}{5+Bi}\right)^{N_4}\cdot\left(\frac{2+Bi}{2}\right)^{N_5}\right]\right\}. \tag{26}$$

3.2 *The second phase* ($\xi_r \leqslant \xi \leqslant +\infty$)

The fundamental relation is also the equation (4) in the form (13), which is now considered in the interval:

$$\xi_r \leqslant \xi \leqslant +\infty; \qquad 0 \leqslant \rho \leqslant 1. \tag{27}$$

The boundary conditions are now:

$$\left[\frac{\partial\theta}{\partial\rho} + Bi\,.\,\theta\right]_{\rho=1} = 0; \tag{28}$$

$$\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0} = 0; \qquad \theta(\rho,\xi_r) = \theta_1. \tag{29}$$

An approximate solution satisfying boundary conditions (29) can be assumed now as follows:

$$\theta_2 = \theta_1 + a_1(\xi) + [a_2(\xi) - a_1(\xi)]\,.\,\rho^2. \tag{30}$$

The function θ_1 describes the temperature distribution at the end of the thermal entrance region ($\xi = \xi_r$, $q(\xi) = 0$). This distribution is the initial distribution for the second phase of the considerations. Its expression can be obtained directly from the relation (19), by substituting $q = 0$.

Thus:

$$\theta_1(\rho,\xi_r) = 1 - \frac{Bi}{2+Bi}\cdot\rho^2. \tag{31}$$

where

$$\left.\begin{aligned}
&A_1 = 5 + Bi; && B_1 = 22 + Bi\,.(8 + Bi); && \\
&A_2 = 4\,.(4 + Bi); && B_2 = 2\,.[50 + 3\,.Bi\,.(7 + Bi)]; && B_6 = 14\,.[2 + Bi\,.(3 + Bi)]; \\
&A_3 = 6\,.(3 + Bi); && B_3 = 2\,.[85 + 7\,.Bi\,.(6 + Bi)]; && B_7 = 2\,.[5 + 7\,.Bi\,.(2 + Bi)]; \\
&A_4 = 4\,.(2 + Bi); && B_4 = 2\,.[60 + 7\,.Bi\,.(5 + Bi)]; && B_8 = 6\,.Bi\,.(1 + Bi); \\
&A_5 = 1 + Bi; && B_5 = 10; && B_9 = Bi^2.
\end{aligned}\right\} \quad (22)$$

The initial condition for equation (21) is:

$$q(\xi)\big|_{\xi=0} = 1. \quad (23)$$

The particular solution of (21) satisfying the initial condition (23) is:

$$\xi = \frac{Pe}{10}.\left\{\frac{N_1}{3}.(1 - q^3) + \frac{N_2}{2}.(1 - q^2) + N_3.(1 - q) + \ln\left(\left[\frac{4}{(5 + Bi) - (1 + Bi).q}\right]^{N_4}.\left[\frac{(2 + Bi) - Bi.q}{2}\right]^{N_5}\right)\right\}; \quad (24)$$

where:

$$\left.\begin{aligned}
N_1 &= -\frac{Bi}{1 + Bi}; \\
N_2 &= \frac{4}{(1 + Bi)^2}; \\
N_3 &= \frac{Bi^4 + 4\,.Bi^3 + 7\,.Bi^2 + 18\,.Bi - 2}{Bi\,.(1 + Bi)^3}; \\
N_4 &= \frac{16\,.Bi^3 + 13\,.Bi^2 + 13\,.Bi + 144}{(1 + Bi)^4\,.(1 - Bi)}; \\
N_5 &= \frac{-4\,.Bi^6 + 16\,.Bi^5 + 57\,.Bi^4 - 19\,.Bi^3 + 100\,.Bi^2 + 32\,.Bi + 4}{Bi^2\,.(1 + Bi)^4\,.(1 - Bi)}.
\end{aligned}\right\} \quad (25)$$

The parameters $a_1(\xi)$ and $a_2(\xi)$ in (30) should satisfy the condition:

$$a_1(\xi_r) = a_2(\xi_r) = 0. \quad (32)$$

Making use of the boundary condition (28) we obtain the relation:

$$a_2(\xi) = a_1(\xi).\frac{2}{2 + Bi}. \quad (33)$$

On substituting (31) and (33) in (30), we obtain a new formulation of the assumed approximate solution, as follows:

$$\theta_2 = [1 + a_1(\xi)].\left(1 - \frac{Bi}{2 + Bi}.\rho^2\right). \quad (34)$$

Similarly to the first phase we have the relation:

$$\int_0^1 \left[\frac{\partial}{\partial \rho}\left(\rho.\frac{\partial \theta_2}{\partial \rho}\right) - \rho.Pe.\frac{\partial \theta_2}{\partial \xi}\right].\mathrm{d}\rho.\delta\theta_2 = 0. \quad (35)$$

On evaluating the above integral, we obtain a differential equation determining the parameter

$a_1(\xi)$. This equation is as follows:

$$\frac{da_1}{d\xi} + P \cdot a_1 = -P; \quad (36)$$

where:

$$P = \frac{6 \cdot Bi \cdot (4 + Bi)}{Pe \cdot (12 + 6 \cdot Bi + Bi^2)}. \quad (37)$$

The general solution of the equation (36) has the form:

$$a_1(\xi) = C \cdot \exp(-P \cdot \xi) - 1; \quad (38)$$

where C is an integration constant to be determined from the condition (32). After evaluation of the integration constant we obtain the relation:

$$a_1(\xi) = -\{1 - \exp[-P \cdot (\xi - \xi_r)]\}; \quad (39)$$

determining the sought—for parameter $a_1(\xi)$. Substitution of (39) into (34) gives the approximate solution for the interval (27). This solution is of the form:

$$\theta_2 = \left(1 - \frac{Bi}{2 + Bi} \cdot \rho^2\right) \cdot \exp[-P \cdot (\xi - \xi_r)]. \quad (40)$$

It satisfies the evident necessary condition:

$$\theta_2(\rho, \xi_r) = \theta_1(\rho, \xi_r). \quad (41)$$

4. NUMERICAL EXAMPLE AND DISCUSSION OF RESULTS

Comparative calculations were made for liquid sodium, assuming for the physical quantities involved, the following numerical values:

thermal conductivity of sodium .... $\lambda = 61{\cdot}2$ [kcal/m . h . °C];

heat-transfer coefficient .... $\alpha = 61{\cdot}2$ [kcal/m^2 . h . °C];

specific heat of sodium .... $c = 0{\cdot}3$ [kcal/kg . °C];

specific weight of sodium .... $\gamma = 854$ [kg/m^3];

flow velocity of sodium .... $w = 3$ [m/s];

outer radius of the pipe .... $R = 0{\cdot}5$ [m];

The computations were carried out by use of an ordinary desk computer on the basis of the expressions (11) and (12) as well as (19), (24), (26) and (40). The values of the roots of the equation (12) were taken from [6]. In the course of calculation process it was found, that it was only in the case of $\xi/\xi_r = 0{\cdot}17$ ($q = 0{\cdot}7$) necessary to use six terms of the series (11). For greater values of ξ four terms were enough, and further even two terms of this series were sufficient, because further terms are very small as compared with the initial ones. The dimensionless length of the thermal entrance region for this example as computed by use of the relation (26) is $\xi_r = 4972$, which under conversion on the dimensional quantity is

$$X_r = \xi_r \cdot R = 4972 \cdot 0{\cdot}5 = 2486 \text{ [m]}.$$

The results of the computation have been assembled in Tables 1–4 and represented graphically in Fig. 3.

The results obtained show that the greatest absolute difference between the exact solution and the approximate one is reached at the end of the thermal entrance region. This difference would probably be smaller, if it was calculated

Table 1. Distribution of the dimensionless temperature parameter θ for $\xi/\xi_r = 0{\cdot}17$ ($q = 0{\cdot}7$)

ρ	θ_{exact}	$\theta_{approx.}$	$\theta_{exact} - \theta_{approx.}$
0	0·9977	1·0000	0·0023
0·35	0·9956	1·0000	0·0044
0·7	0·9989	1·0000	0·0011
0·8	0·9998	0·9923	0·0075
0·9	0·9947	0·9690	0·0257
1·0	0·9641	0·9302	0·0339

thermal diffusivity of sodium ...... $\varkappa = \lambda / c \cdot \gamma = 0{\cdot}239\ m^2/h$;

Biot's number .... $Bi = R \cdot \alpha / \lambda = 0{\cdot}5$;

Péclét's number .... $Pe = R \cdot w / \varkappa = 22600$.

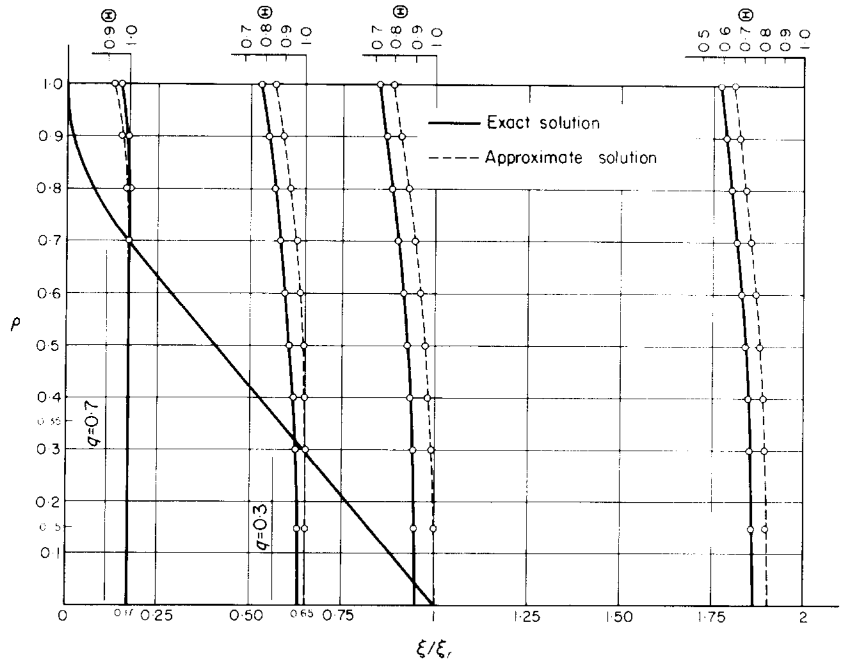

FIG. 3. Graphical representation of the results contained in Tables 1–4.

Table 2. Distribution of the dimensionless temperature parameter θ for $\xi/\xi_r = 0{\cdot}65$ (q = 0·3)

ρ	θ_{exact}	$\theta_{approx.}$	$\theta_{exact} - \theta_{approx.}$
0	0·964	1·000	0·036
0·15	0·961	1·000	0·039
0·3	0·951	1·000	0·049
0·4	0·939	0·997	0·058
0·5	0·924	0·988	0·064
0·6	0·904	0·973	0·069
0·7	0·880	0·951	0·071
0·8	0·852	0·924	0·072
0·9	0·819	0·890	0·071
1·0	0·783	0·851	0·068

Table 3. Distribution of the dimensionless temperature parameter θ for $\xi/\xi_r = 1$ (q′ = 0)

ρ	θ_{exact}	$\theta_{approx.}$	$\theta_{exact} - \theta_{approx.}$
0	0·911	1·000	0·089
0·15	0·907	0·995	0·088
0·3	0·895	0·982	0·087
0·4	0·882	0·968	0·086
0·5	0·865	0·950	0·085
0·6	0·844	0·928	0·084
0·7	0·821	0·902	0·081
0·8	0·793	0·872	0·079
0·9	0·761	0·838	0·077
1·0	0·730	0·800	0·070

Table 4. Distribution of the dimensionless temperature parameter θ *for* $\xi/\xi_r = 2$ $(q = 0)$

ρ	θ_{exact}	$\theta_{approx.}$	$\theta_{exact} - \theta_{approx.}$
0	0·754	0·823	0·069
0·15	0·750	0·819	0·069
0·3	0·739	0·808	0·069
0·4	0·728	0·796	0·068
0·5	0·713	0·782	0·069
0·6	0·695	0·764	0·069
0·7	0·674	0·742	0·068
0·8	0·651	0·717	0·066
0·9	0·625	0·689	0·064
1·0	0·597	0·658	0·061

using more terms of the exact solution (relations (11) and (12)). However, this would require time-absorbing (and expensive) computations by use of a digital computer. It seems not to be necessary to carry out such a work, our object being rather of a qualitative nature.

An interesting fact to notice is, that the exact and the approximate solution are nearly parallel in particular for greater values of the variable ξ (Fig. 3). It should also be noticed, that the absolute difference between the two solutions obtained decreases with increasing ξ, because the exact solution (11) and the approximate one (40) tend both to zero for ξ tending to infinity. The comparison of the exact and the approximate solution in the thermal entrance region (bounded by the radius $q(\xi)$) confirms the validity of the interpretation of this region as a region where the temperature is constant (or nearly constant) and it varies outside this region only. The values of the temperature obtained for the thermal entrance region indicate much greater variability in the direction of ρ than in the direction of ξ.

It is also noteworthy, that the lengths of the thermal entrance region can be considerable. In the particular example computated here and concerning a pipe of 1 m dia. the temperature penetration has reached the axis of the flow only at the distance of 2486 m from the origin.

The above confrontation of the interpretation used in the approximate solution process for the thermal entrance region with the exact solution can be treated as a basis for application of the restricted variation principle in cases, in which the exact solution is unknown.

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APPENDIX

Evaluation of the inverse Laplace transform for $\bar{\theta}(\rho, s)$

The Laplace transform under consideration is:

$$\bar{\theta}(\rho, s) = \frac{1}{s} \times \left[1 - \frac{Bi\, I J_0(\sqrt{(Pe\,.\,s)}\,.\,\rho)}{\sqrt{(Pe.s)}.I_1[\sqrt{(Pe.s)}] + Bi.I_0(\sqrt{(Pe.s)})}\right]. \tag{A.1}$$

The inverse Laplace transform of (A.1) can be written as follows:

$$\theta(\rho,\xi) = \mathscr{L}^{-1}[\bar{\theta}(\rho,s)]$$
$$= 1 - \mathscr{L}^{-1}\left[\frac{1}{s}\cdot\frac{Bi\,.\,I_0[\sqrt{(Pe\,.\,s)}\,.\,\rho]}{\sqrt{(Pe\,.\,s)}\,.\,I_1[\sqrt{(Pe\,.\,s)}] + Bi\,.\,I_0[\sqrt{(Pe\,.\,s)}]}\right]$$
$$= 1 - \mathscr{L}^{-1}\left[\frac{P(s)}{s\,.\,Q(s)}\right]; \quad \text{(A.2)}$$

where

$$P(s) = Bi\,.\,I_0(\sqrt{(Pe\,.\,s)}\,.\,\rho); \quad \text{(A.3)}$$

$$Q(s) = \sqrt{(Pe\,.\,s)}\,.\,I_1[\sqrt{(Pe\,.\,s)}] + Bi\,.\,I_0[\sqrt{(Pe\,.\,s)}]. \quad \text{(A.4)}$$

For the last term of (A.2) there exists the known relation [8]:

$$\mathscr{L}^{-1}\left[\frac{P(s)}{s\,.\,Q(s)}\right] = \frac{P(0)}{Q(0)}$$
$$+\sum_{n=1}^{\infty}\frac{P(s_n)}{[s\,.\,\mathrm{d}Q/\mathrm{d}s]_{s=s_n}}\,.\,\exp(s_n\,.\,\xi); \quad \text{(A.5)}$$

in which the quantities s_n are the roots of the expression (A.4).

According to (A.3) and (A.4) we have:

$$P(s)|_{s=0} = Bi; \qquad Q(s)|_{s=0} = Bi. \quad \text{(A.6)}$$

Substituting (A.6) in (A.5) and then in (A.2), we obtain:

$$\theta(\rho,\xi) = -\sum_{n=1}^{\infty}\frac{P(s_n)}{[s\,.\,\mathrm{d}Q/\mathrm{d}s]_{s=s_n}}\,.\,\exp(s_n\,.\,\xi). \quad \text{(A.7)}$$

Using the auxiliary notation for $(Pe\,.\,s)$, we can write the relation (A.4) in the form:

$$Q(s) = m\,.\,I_1(m) + Bi\,.\,I_0(m); \quad \text{(A.8)}$$

and, making use of the properties of Bessel functions [9], transform it as follows:

$$Q(s) = \frac{m}{i}\,.\,J_1(i\,.\,m) + Bi\,.\,J_0(i\,.\,m); \quad \text{(A.9)}$$

where: $i = \sqrt{-1}$.

On introducing the notation:

$$i\,.\,m = \zeta; \quad \text{(A.10)}$$

we can write (A.9) in the form:

$$Q(s) = Bi\,.\,J_0(\zeta) - \zeta\,.\,J_1(\zeta) = Q(\zeta). \quad \text{(A.11)}$$

Making use of (A.10) in the auxiliary notation for $(Pe\,.\,s)$ we obtain:

$$s = -\frac{\zeta^2}{Pe}. \quad \text{(A.12)}$$

From (A.11) it follows, that there is an infinite number of roots $s_n = -\zeta_n^2/Pe$, satisfying the condition:

$$\zeta_n\,.\,J_1(\zeta_n) - Bi\,.\,J_0(\zeta_n) = 0. \quad \text{(A.13)}$$

The first six roots of the above equation were found in [6].

The denominator of the relation (A.7) can be evaluated as follows:

$$\left[s\,.\,\frac{\mathrm{d}Q}{\mathrm{d}s}\right]_{s=s_n} = \left\{s\,.\,\frac{\mathrm{d}}{\mathrm{d}s}\left[\sqrt{(Pe\,.\,s)}\,.\,I_1[\sqrt{(Pe\,.\,s)}]\right.\right.$$
$$\left.\left.+\,Bi\,.\,I_0(\sqrt{(Pe\,.\,s)})\right]\right\}_{s=s_n} =$$
$$= \tfrac{1}{2}\,.\,[(Pe\,.\,s)\,.\,I_0(\sqrt{(Pe\,.\,s)}) + Bi\,.\,\sqrt{(Pe\,.\,s)}\,.\,I_1(\sqrt{(Pe\,.\,s)})]_{s=s_n}$$
$$= \tfrac{1}{2}\,.\,[m^2\,.\,I_0(m) + Bi\,.\,m\,.\,I_1(m)]_{s=s_n}$$
$$= \tfrac{1}{2}\,.\,[m^2\,.\,J_0(i\,.\,m) - Bi\,.\,i\,.\,m\,.\,J_1(i\,.\,m)]_{s=s_n}$$
$$= \tfrac{1}{2}[-\zeta^2\,.\,J_0(\zeta) - Bi\,.\,\zeta\,.\,J_1(\zeta)]_{\zeta=\zeta_n}$$
$$= -\tfrac{1}{2}\,.\,[\zeta_n^2\,.\,J_0(\zeta_n) + Bi\,.\,\zeta_n\,.\,J_1(\zeta_n)]. \quad \text{(A.14)}$$

According to (A.13) we have:

$$\zeta_n\,.\,J_1(\zeta_n) = Bi\,.\,J_0(\zeta_n). \quad \text{(A.15)}$$

Therefore

$$\left[s\,.\,\frac{\mathrm{d}Q}{\mathrm{d}s}\right]_{s=s_n} = -\tfrac{1}{2}\,.\,(Bi^2 + \zeta_n^2)\,.\,J_0(\zeta_n). \quad \text{(A.16)}$$

The expression (A.3) can also be transformed in the same manner:

$$P(s) = Bi\,.\,I_0[\sqrt{(Pe\,.\,s)}\,.\,\rho] = Bi\,.\,I_0(m\,.\,\rho)$$
$$= Bi\,.\,J_0(i\,.\,m\,.\,\rho) = Bi\,.\,J_0(\zeta\,.\,\rho). \quad \text{(A.17)}$$

Therefore

$$P(s_n) = Bi\,.\,J_0(\zeta_n\,.\,\rho). \quad \text{(A.18)}$$

Substituting (A.18) and (A.16) into (A.7), we obtain the inverse transform in the form:

$$\theta(\rho,\xi) = 2.\,Bi\,.\sum_{n=1}^{\infty}\frac{J_0(\zeta_n\,.\,\rho)}{(Bi^2+\zeta_n^2)\,.\,J_0(\zeta_n)}\,.\,\exp\left(-\frac{\zeta_n^2}{Pe}\,.\,\xi\right). \quad \text{(A.19)}$$

RECHERCHE THÉORIQUE SUR LA RÉGION D'ENTRÉE THERMIQUE POUR UN ÉCOULEMENT STATIONNAIRE À SYMÉTRIE AXIALE AVEC DES CONDITIONS AUX LIMITES MIXTES

Résumé—Le but du présent travail est celui de résoudre, d'une methode approximatique, le probléme de la couche-limite thermique et du champ des temperatures dans un écoulement stationnaire à symmetrie axiale et à vitesse uniforme (rod-like flow) d'un liquide incompressible. Les quantites caractérisant les propriétés du milieu sont traités comme constantes. En outre ou admet que la chaleur ne peut pas être produite dans le milieu. Ce problême est resolu d'une manière exacte- au moyen de la transformation de Laplace. Pour comparaison, il est résolu ègalement au moyen de la méthode de variation restreinte. Pour illustrer ces considérations ou donne un exemple numerique, dans lesquel la valeur du nombre de Biot est $Bi = 0{\cdot}5$ et celle de Péclét$Pe = 22600$. Les résultats obtenus permettens de conclure que la méthode de la variation restreinte est applicable à ceux pour lesquels la solution exacte n'est pas connue.

THEORETISCHE UNTERSUCHUNG ÜBER DEN THERMISCHEN EINLAUFBEREICH BEI STATIONÄRER ACHSIAL SYMMETRISCHER KOLBENSTRÖMUNG MIT GEMISCHTEN RANDBEDINGUNGEN

Zusammenfassung—Die vorliegende Arbeit beschäftigt sich mit der angenäherten Lösung der thermischen Grenzschicht sowie mit dem Temperaturfeld in der axialsymmetrischen, stationären Kolbenströmung einer inkompressiblen Flüssigkeit. Die physikalischen Eigenschaften des Mediums werden als Konstant angenommen. Hinzu kommt noch, dass im Medium selbst die innere Wärme nicht entwickelt werden kann. Es wurde auch eine exakte Lösung dieses Problems mit Hilfe der Laplace schen Transformation gefunden. Des Vergleichszwecks wegen wurde dasselbe Problem mit der Methode der begrenzten Variation näherungsweise gelöst. Zur Erläuterung der Theorie wird ein Beispiel für Biot'sche Zahl $Bi = 0{\cdot}5$ und für Péclét'sche Zahl $Pe = 22600$ berechnet. Die gewonnenen Ergebnisse ermöglichen den Schluss über die Anwendbarkeit der Methode der begrenzten Variation für den Fall, in dem keine exakte Lösung gefunden sein kann, zu ziehen.

ТЕОРЕТИЧЕСКОЕ ИССЛЕДОВАНИЕ ВХОДНОГО УЧАСТКА СТАЦИОНАРНОГО ОСЕСИММЕТРИЧНОГО СТЕРЖНЕВОГО ТЕЧЕНИЯ ПРИ СМЕШАННЫХ ГРАНИЧНЫХ УСЛОВИЯХ

Аннотация—В работе проводится теоретический анализ роста теплового пограничного слоя и теплообмена в стационарном осесимметричном стержневом течении несжимаемой жидкости с постоянными физическими свойствами при отсутствии внутренних источников тепла с граничными условиями третьего рода. Получено точное решение задачи с помощью преобразований Лапласа. Для сравнения проведено приолиженное решение этой задачи с использованием вариационных методов. Рассчитано температурное поле в потоке натрия для случая критерия Био $Bi = 0{,}5$ и критерия Пекле $Pe = 22600$. Для расчета использовались точное и приолиженное решения. Полученные результаты позволили сделать вывод о возможности применения вариационных методов для случаев, в которых невозможно найти точное решение.

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